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A PROOF OF YOMDIN-GROMOV'S ALGEBRAIC LEMMA

BY

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ABSTRACT

Following the analysis of differentiable mappings of Y. Yomdin, M. Gromov has stated a very elegant "Algebraic Lemma" which says that the "differentiable size" of an algebraic subset may be bounded only in terms of its dimension, degree and diameter, regardless of the size and specific values of the underlying coefficients. We give a complete and elementary proof of Gromov's result.

1. Introduction

A semi-algebraic set is a subset of some $\mathbb{R}^d$ defined by a finite number of polynomial inequalities and equalities. Its degree is the sum of the total degrees of the polynomials involved. A semi-algebraic map is a map whose graph is a semi-algebraic set and the degree of the map is the degree of the set. Necessary definitions and basic properties of real semi-algebraic geometry are recalled in the third section.

Y. Yomdin [8] developed many tools around "quantitative Sard Lemmas" involving the differentiable size of semi-algebraic sets. M. Gromov observed that one of these tools could be refined to give the following very elegant statement:

THEOREM 1 (Yomdin-Gromov's algebraic Lemma): *Let r, l and d be positive integers. For any semi-algebraic compact subset $A \subset [0,1]^d$ of dimension l, there exist an integer N and continuous semi-algebraic maps $\phi_1, \dots, \phi_N : [0,1]^l \to [0,1]^d$, such that:*

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- *ϕ_i is analytic on $]0,1[^l$;*
- *$\|\phi_i\|_r := \max_{\beta:|\beta|\leq r} \|\partial^\beta \phi_{i/]0,1[^l}\|_\infty \leq 1$;*
- *$\bigcup_{i=0}^N \phi_i([0,1]^l) = A$.*

Moreover N and $deg(\phi_i)$ are bounded by a function of $deg(A)$, d and r.

In his Séminaire Bourbaki [12], M. Gromov gives many ideas but stops short of a complete proof. In [15], [16], Y. Yomdin used a weaker version of the previous theorem. In this initial form, the parametrizations omitted a subset covered with at most $Clog(1/\alpha)$ cubes of radius α, for arbitrarily small $\alpha > 0$. This version was sufficient for the dynamical applications presented in [15], [16].

By using polynomial Taylor's approximation, this theorem gives estimates of the local complexity of smooth maps. Yomdin used it to compare the topological entropy and the "homological size" for $\mathcal{C}^r$ maps. S. Newhouse [13] showed, using Pesin's theory, how this gives, for $\mathcal{C}^\infty$ smooth maps, upper-semicontinuity of the metric entropy and therefore the existence of invariant measures with maximum entropy. J. Buzzi [6] observed that, in fact, Y. Yomdin's estimates give a more uniform result called asymptotic h-expansiveness, which was shown by M. Boyle, D. Fiebig and U. Fiebig [3] to be equivalent to the existence of principal symbolic extensions for $\mathcal{C}^\infty$ smooth maps. The dynamical consequences of the above theorem are still developing in the works of M. Boyle, T. Downarowicz, S. Newhouse and others [11], [4].

The theorem is trivial for $d = 1$: the semi-algebraic subsets of $[0,1]$ are the finite unions of subintervals of $[0,1]$. We deal with the 2-dimensional case as suggested by M. Gromov. This simple and instructive case is the subject of Section 5. We prove the general case by induction using the notion of $(\mathcal{C}^\alpha, K)$ triangular maps introduced in Section 2. The induction steps are of three types:

- we consider a semi-algebraic map defined on a semi-algebraic set of higher dimension and we bind the first derivative in the first coordinate.
- fixing the dimension of the semi-algebraic set, we bind the derivatives of the next higher order with respect to the first coordinate.
- fixing the dimension of the semi-algebraic set and the order of derivation, we bind the next partial derivative for a total order on $\mathbb{N}^d$.

As I was completing the submission of this paper, I learned that Pila and A. Wilkie had written a proof of the same theorem [14]. I am grateful to M. Coste for this reference.

2. Technical tools

Yomdin-Gromov's Lemma is proved by controlling the derivatives consecutively. This is possible by the notion of triangular maps, which already appears in Y. Yomdin's works [16].

2.1. Triangular maps.

Definition 1: A map $\psi :]0,1[^l \to]0,1[^d$ is triangular if $l \leq d$ and if

$$\psi = (\psi_1(x_1,\dots,x_l),\dots,\psi_{d-l+1}(x_1,\dots,x_l),$$
$$\psi_{d-l+2}(x_2,\dots,x_l),\dots,\psi_{d-l+k}(x_k,\dots,x_l),\dots,\psi_d(x_l)),$$

for a family of maps $(\psi_i :]0,1[^{\min(l,d+1-i)} \to]0,1[)_{i=1,\dots,d}$.

Fact 1: If $\psi :]0,1[^m \to]0,1[^p$ and $\phi :]0,1[^n \to]0,1[^m$ are triangular, then so is $\psi \circ \phi :]0,1[^n \to]0,1[^p$.

2.2. $(\mathcal{C}^\alpha, K)$ maps. First we introduce the order on $\mathbb{N}^d$ used for the induction in the proof of Yomdin-Gromov's algebraic lemma.

Definition 2: $\mathbb{N}^d$ is endowed with the order $\preceq$, defined as follows: for $\alpha = (\alpha_1,\dots,\alpha_d)$, $\beta = (\beta_1,\dots,\beta_d) \in \mathbb{N}^d$, $\alpha \preceq \beta$ if $(\alpha = \beta)$ either $(|\alpha| := \sum_i \alpha_i < |\beta|)$ or ($|\alpha| = |\beta|$ and $\alpha_k \leq \beta_k$, where $k := \max\{l \leq d \ : \ \alpha_l \neq \beta_l\}$).

In fact, we have $\alpha \preceq \beta$ if and only if $(|\alpha|, \alpha_d, \alpha_{d-1},\dots,\alpha_1)$ precedes $(|\beta|, \beta_d, \beta_{d-1},\dots,\beta_1)$ in the usual lexicographic order.

Definition 3: Let $K \in \mathbb{R}^+$, $\alpha \in \mathbb{N}^d - \{0\}$. Let $A \subset]0,1[^d$ be an open set. A map $f : A \to \mathbb{R}^k$ is a $\boldsymbol{(\mathcal{C}^\alpha, K)}$ **map**, if $f := (f_1,\dots,f_k)$ is a $\mathcal{C}^{|\alpha|}$ map and if $\|f\|_\alpha := \max_{\beta \preceq \alpha, 1 \leq i \leq k} \|\partial^\beta f_i\|_\infty \leq K$.

If $\alpha = (0,0\dots,0,r)$ (i.e., all the partial derivatives of $f$of order up to r are bounded by K), we write $(\mathcal{C}^r, K)$ and $\|.\|_r$ instead of $(\mathcal{C}^{(0,\dots,0,r)}, K)$ and $\|\cdot\|_{(0,\dots,0,r)}$.

2.3. Composition of $(\mathcal{C}^\alpha, 1)$ maps. The two following lemmas deal with the composition of $(\mathcal{C}^\alpha, 1)$ maps.

Lemma 1: *For all $d, r \in \mathbb{N}^*$, there exists a real number $K = K(d,r)$, such that if $\psi, \phi :]0,1[^d \to]0,1[^d$ are two $(\mathcal{C}^r, 1)$ maps, then $\psi \circ \phi$ is a $(\mathcal{C}^r, K)$ map.*

Proof. It follows directly from the formula of Faa-di-bruno for the higher derivatives of a composition (See [1, p. 3]), which we recall for completeness: let $\psi, \phi :]0,1[^d \to]0,1[^d$ be two $\mathcal{C}^r$ maps and let $(h_1, \ldots, h_r) \in (\mathbb{R}^d)^r$, we have [2]:

$$D^r(\psi \circ \phi)(x)(h_1, \ldots, h_r) = \sum_{1 \le q \le r} \sum_{i_1, \ldots, i_q} \sigma_q(i_1, \ldots, i_q) \\ \times D^q\psi(\phi(x))(D^{i_1}\phi(x)(h_1, \ldots, h_{i_1}), \ldots, D^{i_q}\phi(x)(h_{r-i_q+1}, \ldots, h_r)),$$

where the second sum is over all nonzero integers $i_1, \ldots, i_q$ satisfying

$$\sum_{k=1}^{q} i_k = r. \qquad \blacksquare$$

We shall need the following adaptation of Lemma 1 to triangular maps.

LEMMA 2: *For all $d, r \in \mathbb{N}^*$, there exists a real $K = K(r,d)$ such that if $\psi, \phi :]0,1[^d \to]0,1[^d$ are two $(\mathcal{C}^\alpha, 1)$ maps with $|\alpha| = r$ and if ϕ is a triangular map, then $\psi \circ \phi$ is a $(\mathcal{C}^\alpha, K)$ map.*

We introduce some notation for the proof of Lemma 2. Let $(e_i)_{i=1,\ldots,d}$ be the canonical basis of $\mathbb{R}^d$. For $i = 1, \ldots, d$, $V_i \subset \mathbb{R}^d$ is the vector space generated by $e_1, \ldots, e_i$.

For $\alpha \in \mathbb{N}^d$ with $|\alpha| = r$, $v^\alpha := (\underbrace{e_1, \ldots, e_1}_{\alpha_1}, \ldots, \underbrace{e_d, \ldots, e_d}_{\alpha_d}) \in (\mathbb{R}^d)^r$ and $V^\alpha := \underbrace{V_1 \times \cdots \times V_1}_{\alpha_1} \times \cdots \times \underbrace{V_d \times \cdots \times V_d}_{\alpha_d} \subset (\mathbb{R}^d)^r$. Observe that, for a $\mathcal{C}^r$ map $f :]0,1[^d \to]0,1[^d$, we have $\partial_\alpha f(x) = D^r f(x)(v^\alpha)$.

Fact 2: Let $1 \le k \le d$ be an integer. Let $f := (f_1, \ldots, f_d) :]0,1[^d \to]0,1[^d$ be a $\mathcal{C}^1$ triangular map. Then for all $x \in]0,1[^d$, $\partial_{x_k} f(x) \in V_k$.

Proof. Let $l > k$ be an integer. The map f being triangular,

$$f_l(x) = f_l(x_l, \ldots, x_d)$$

and therefore we have $\partial_{x_k} f_l = 0$. $\blacksquare$

Fact 3: Let $f :]0,1[^d \to]0,1[^d$ be a $\mathcal{C}^r$ map. Let $\alpha \in \mathbb{N}^d$ with $|\alpha| = r$ and $w \in V^\alpha$. Then $D^r f(x)(w) = \sum_{\beta \preceq \alpha} w_\beta \partial_\beta f(x)$, where w_β is a polynomial in the coordinates of w, of which the coefficients depend only on r and d. If

[2] $\sigma_q(i_1, \ldots, i_q) = \prod_{0 \le k \le q-1} \binom{r-1-\sum_{l=1}^{k} i_l}{r-\sum_{l=1}^{k+1} i_l}$

$w := (v_1, \dots, v_d) \in (\mathbb{R}^d)^r$, we have $w_\alpha = \prod_{i=1,\dots,d} v_{i,i}^{\alpha_i}$, where $v_{i,i}$ denotes the i^{th} coordinate of v_i.

Proof. For

$$w \in \underbrace{\{e_1\} \times \cdots \times \{e_1\}}_{\alpha_1} \times \cdots \times \underbrace{\{e_1, e_2, \dots, e_d\} \times \cdots \times \{e_1, e_2, \dots, e_d\}}_{\alpha_d} \subset V_\alpha,$$

it follows from the definition of the order $\preceq$. We conclude the proof by multilinearity. ∎

Lemma 2 is easily implied by the following

Fact 4: Let $\alpha \in \mathbb{N}^d$ with $|\alpha| = r$. Let $\psi, \phi :]0,1[^d \to]0,1[^d$ be two $\mathcal{C}^r$ maps. We assume also, that ϕ is a triangular map. Then

$$\partial_\alpha(\psi \circ \phi)(x) = \partial_\alpha \psi(\phi(x)) \prod_{i=1,\dots,d} (\partial_{x_i} \phi_i(x))^{\alpha_i} + R(\partial_\beta \psi, \partial_\gamma \phi : \beta \prec {}^1 \alpha, \gamma \preceq \alpha),$$

where R is a polynomial depending only on r and d.

Proof. Let $(v_1, \dots, v_d) := v_\alpha$. Using the formula of Faa-di-bruno, we only have to consider the general term

$$D^q \psi(\phi(x))(D^{i_1} \phi(x)(v_1, \dots, v_{i_1}), \dots, D^{i_q} \phi(x)(v_{r-i_q+1}, \dots, v_r))$$

for some nonzero integers $i_1, \dots, i_q$ satisfying $\sum_{k=1}^q i_k = r$. We have only to study the two following cases (only derivatives of ψ and ϕ of order $< r$ are involved in the other terms):

- $q = 1$ and $i_1 = r > 1$: the corresponding term is

  $$D\psi(\phi(x))(D^r \phi(x)(v)) = D\psi(\phi(x))(\partial_\alpha \phi(x)).$$

  Therefore, this term contains also only derivatives of ψ of order $\prec \alpha$ and derivatives of ϕ of order $\preceq \alpha$.
- $q = r$ and $i_1 = i_2 = \cdots = i_q = 1$: the corresponding term is

  $$D\psi(\phi(x))(\underbrace{\partial_{x_1}\phi, \dots, \partial_{x_1}}_{\alpha_1}\phi, \dots, \underbrace{\partial_{x_d}\phi, \dots, \partial_{x_d}\phi}_{\alpha_d}).$$

  By Fact 2, $(\underbrace{\partial_{x_1}\phi, \dots, \partial_{x_1}}_{\alpha_1}\phi, \dots, \underbrace{\partial_{x_d}\phi, \dots, \partial_{x_d}\phi}_{\alpha_d}) \in V^\alpha$. Then we apply Fact 3 to get the desired result. ∎

[1] Let $\alpha, \beta \in \mathbb{N}^d$, $\beta \prec \alpha$ if and only if $\beta \preceq \alpha$ and $\beta \neq \alpha$.

3. Real semi-algebraic geometry

In this section we recall basic results concerning semi-algebraic sets. We borrow them from [2], [7] and [9].

3.1. Semi-algebraic sets and maps.

Definition 4: $A \subset \mathbb{R}^d$ is a **semi-algebraic set** if it can be written as a finite union of sets of the form

$$\{x \in \mathbb{R}^d \colon P_1(x) > 0, \ldots, P_r(x) > 0, P_{r+1}(x) = 0, \ldots, P_{r+s}(x) = 0\},$$

where $r, s \in \mathbb{N}$ and $P_1, \ldots, P_{r+s} \in \mathbb{R}[X_1, \ldots, X_d]$. Such a formula is called a **presentation** of A.

The degree of a presentation is the sum of the total degrees of the polynomials involved (with multiplicities). The **degree** $deg(A)$ **of a semi-algebraic set** A is the minimum degree of its presentations.

Remark that the number of polynomials occurring in a presentation of a semi-algebraic set is bounded by the degree of this presentation.

Definition 5: $f : A \subset \mathbb{R}^d \to \mathbb{R}^n$ is a **semi-algebraic map** if the graph $\Gamma_f := \{(x, f(x)) : x \in A\} \subset \mathbb{R}^d \times \mathbb{R}^n$ of f is a semi-algebraic set. The **degree** $deg(f)$ **of a semi-algebraic map** f is the degree of its graph Γ_f.

Definition 6: A **Nash manifold** is a real analytic submanifold of $\mathbb{R}^d$, which is also a semi-algebraic set.

A **Nash map** is a map defined on a Nash manifold, which is both analytic and semi-algebraic.

3.2. Tarski's Principle.

Theorem 2 (Tarski's principle): *Let $A \subset \mathbb{R}^{d+1}$ be a semi-algebraic set and $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ the projection defined by $\pi(x_1, \ldots, x_{d+1}) = (x_1, \ldots, x_d)$, then $\pi(A)$ is a semi-algebraic set and $deg(\pi(A))$ is bounded by a function of $deg(A)$ and d.*

Proof. See [7, Theorem 2.2.1]. ∎

Corollary 1: *Any formula combining sign conditions on semi-algebraic functions by conjunction, disjunction, negation and universal and existential real* *quantifiers defines a semi-algebraic set. Moreover the degree of this semi-algebraic set is bounded by a function of the degrees of the semi-algebraic functions involved in the formula.*

Proof. See [7, Proposition 2.2.4]. ∎

COROLLARY 2: *Let $f : A \subset \mathbb{R}^d \to \mathbb{R}^n$ be a semi-algebraic map, then A and $f(A)$ are semi-algebraic sets. Moreover, $deg(A)$ and $deg(f(A))$ are bounded by a function of $deg(f), d$ and n.*

Proof. Immediate. ∎

COROLLARY 3: *If ϕ and ψ are two semi-algebraic maps, such that the composition $\phi \circ \psi$ is well-defined, then $\phi \circ \psi$ is a semi-algebraic map. Moreover, its degree is bounded by a function of $deg(\phi)$ and $deg(\psi)$.*

Proof. See [7, Proposition 2.2.6]. ∎

COROLLARY 4: *Let $r \in \mathbb{N}$. Let $A \subset \mathbb{R}^d$ be a semi-algebraic open set and let $f : A \to \mathbb{R}^n$ be a Nash map. The partial derivatives of f of order r are also semi-algebraic maps of degree bounded by a function of $deg(f)$, d, n and r.*

Proof. See [7, Proposition 2.9.1]. ∎

3.3. CONTINUOUS STRUCTURE OF SEMI-ALGEBRAIC SETS. We recall now classical results concerning the structure of semi-algebraic sets. The first results deal with stratification and the last ones with decomposition into cells.

PROPOSITION 1: *For any semi-algebraic subset $A \subset]0,1[^{d+1}$, there exist integers m, $q_1, \ldots, q_m$, disjoint Nash manifolds $A_1, \ldots, A_m \subset]0,1[^d$ and Nash maps, $\zeta_{i,1} < \cdots < \zeta_{i,q_i} : A_i \to]0,1[$, for all $1 \leq i \leq m$, such that:*

- *A coincides with a union of slices of the following two forms $\{(x_1, y) \in]0,1[\times A_i : \zeta_{i,k}(y) < x_1 < \zeta_{i,k+1}(y)\}$ and $\{(\zeta_{i,k}(y), y) : y \in A_i\}$;*
- *the integers m, q_i, $deg(A_i)$, $deg(\zeta_{i,j})$ are bounded by a function of $deg(A)$ and d.*

Proof. This is Theorem 2.2.1 in [2] except that there the maps $\zeta_{i,k}$ are only claimed to be continuous. Using Thom's Lemma, we can assume, that $\zeta_{i,k}$ are Nash maps, as noticed in Remark 1 of [9]. ∎

We let $\mathrm{adh}(H)$, $\mathrm{int}(H)$ and ∂H denote the closure, the interior and the boundary, respectively, of the set $H \subset \mathbb{R}^d$ for the usual topology.

For open semi-algebraic sets, we have the following result.

Corollary 5: *For any semi-algebraic open subset $A \subset]0,1[^{d+1}$, there exist integers m, $q_1, \dots, q_m$, disjoint semi-algebraic open sets $A_1, \dots, A_m \subset]0,1[^d$ and Nash maps, $\zeta_{i,1} < \cdots < \zeta_{i,q_i} : A_i \to]0,1[$, for all $1 \leq i \leq m$, such that:*

- $\mathrm{adh}(A)$ *coincides with a union of slices of the following form*

$$\mathrm{adh}(\{(x_1, y) \in]0,1[\times A_i : \zeta_{i,k}(y) < x_1 < \zeta_{i,k+1}(y)\});$$

- *the integers m, q_i, $deg(A_i)$, $deg(\zeta_{i,j})$ are bounded by a function of $deg(A)$ and d.*

Proof. Let $A \subset]0,1[^{d+1}$ be a semi-algebraic open set. We apply Proposition 1 to A, and keep only the slices of the form

$$\{(x_1, y) \in]0,1[\times A_i : \zeta_{i,k}(y) < x_1 < \zeta_{i,k+1}(y)\},$$

where A_i is an open set. Let us check that the closure of these slices is $\mathrm{adh}(A)$. Let $x \in A$ and let $U \subset A$ be an open neighborhood of x. If the dimension of A_i is strictly less than d, then the slices $\{(\zeta_{i,k}(y), y) : y \in A_i\}$ and $\{(x_1, y) \in]0,1[\times A_i : \zeta_{i,k}(y) < x_1 < \zeta_{i,k+1}(y)\}$ have empty interior. Therefore the open set $U \subset A$ cannot intersect only such slices. We conclude that $x \in \bigcup_{\{i \,:\, A_i \text{is open}\}} \mathrm{adh}(\{(x_1, y) \in]0,1[\times A_i : \zeta_{i,k}(y) < x_1 < \zeta_{i,k+1}(y)\})$, and then $A \subset \bigcup_{\{i \,:\, A_i \text{is open}\}} \mathrm{adh}(\{(x_1, y) \in]0,1[\times A_i : \zeta_{i,k}(y) < x_1 < \zeta_{i,k+1}(y)\})$. ∎

Proposition 2: *Let $A \subset \mathbb{R}^n$ be a semi-algebraic set. There exist an integer N bounded by a function of $deg(A)$ and connected Nash manifolds $A_1, \dots, A_N$ such that $A = \coprod_{i=1}^N A_i$ and $\forall i \neq j$ $(A_i \bigcap \mathrm{adh}(A_j) \neq \emptyset) \Rightarrow (A_i \subset \mathrm{adh}(A_j)$ et $dim(A_i) < dim(A_j))$. ($\coprod$: disjoint union).*

Proof. See [9, Proposition 3.5, p. 124]. ∎

Definition 7: In the notation of the previous proposition, the **dimension** of A is the maximum of the dimensions of the Nash manifolds $A_1, \dots, A_N$.

In the following corollary, we reparametrize a semi-algebraic set with Nash maps of bounded degree. The point of Yomdin-Gromov's algebraic lemma is that one can bound the differentiable size of the reparametrizations. The corollary 6 is a stronger form of Theorem 2.3.6 in [7], so we produce a detailed proof.

Definition 8: Let $A \subset]0,1[^d$ be a semi-algebraic set of dimension l. A family of maps $(\phi_i :]0,1[^l \to A)_{i=1,\dots,N}$ is a **resolution** of A if:

- each ϕ_i is triangular;
- each ϕ_i is a Nash map;
- $A = \bigcup_{i=1}^{N} \phi_i(]0,1[^l)$ [2].

Let $M \in \mathbb{N}$. A M**-resolution** of A is a resolution of A, $(\phi_i)_{i=1,\dots,N}$, such that:

- $N \leq M$;
- $deg(\phi_i) \leq M$.

Any semi-algebraic set $A \subset]0,1[^d$ admits a resolution, $(\phi_i)_{i=1,\dots,N}$, with N and $deg(\phi_i)$ bounded by a function of $deg(A)$ and d. In a formal way:

COROLLARY 6: *Given integers d, δ, there exists an integer $M = M(d,\delta)$, such that any semi-algebraic set $A \subset]0,1[^d$ of degree $\leq \delta$ admits a M-resolution.*

Proof. We argue by induction on d. We denote $P(d)$ the claim of the above corollary for semi-algebraic subsets of $]0,1[^d$. $P(0)$ is trivial. Assume $P(d)$.

Let $A \subset]0,1[^{d+1}$ be a semi-algebraic set of dimension l. Proposition 1 gives us integers m, $q_1,\dots,q_m$, disjoint Nash manifolds $A_1,\dots,A_m \subset]0,1[^d$ and Nash maps, $\zeta_{i,1} < \dots < \zeta_{i,q_i} : A_i \to]0,1[$ such that:

- A coincides with a union of slices of the two following forms $\{(x_1,y) \in]0,1[\times A_i : \zeta_{i,k}(y) < x_1 < \zeta_{i,k+1}(y)\}$ and $\{(\zeta_{i,k}(y),y) : y \in A_i\}$;
- m, q_i, $deg(A_i)$, $deg(\zeta_{i,j})$ are bounded by a function of $deg(A)$ and d.

We note l_i the dimension of A_i; we have: $l_i \leq l$. Apply the induction hypothesis to $A_i \subset]0,1[^d$. There exists a resolution of A_i, i.e., an integer N_i and Nash maps $\phi_{i,1},\dots,\phi_{i,N_i} :]0,1[^{l_i} \to A_i$, such that $A_i = \bigcup_{p=1}^{N_i} \phi_{i,p}(]0,1[^{l_i})$ and N_i, $deg(\phi_i)$ are bounded by a function of $deg(A_i)$ and d, therefore by a function of $deg(A)$ and d.

First, we consider a slice of the form

$$\{(x_1,y) \in]0,1[\times A_i : \zeta_{i,k}(y) < x_1 < \zeta_{i,k+1}(y)\}$$

[2] by convention $]0,1[^0 = \{0\}$.

Observe that the dimension l_i of the Nash manifold A_i is, in this case, strictly less than l. Then, we define $\psi_{i,k,p} :]0,1[^l \to A$ as follows: $\psi_{i,k,p}(x_1, x_2, \ldots, x_l) := (x_1(\zeta_{i,k+1} - \zeta_{i,k}) \circ \phi_{i,p}(x_2, \ldots, x_{l_i+1}) + \zeta_{i,k} \circ \phi_{i,p}(x_2, \ldots, x_{l_i+1}), \phi_{i,p})$, for $1 \leq p \leq N_i$.

Consider now a slice of the form $\{(\zeta_{i,k}(y), y) : y \in A_i\}$. We define

$$\psi_{i,k,p} :]0,1[^l \to]0,1[^{d+1}$$

as follows: $\psi_{i,k,p}(x_1, \ldots, x_l) := (\zeta_{i,k} \circ \phi_{i,p}(x_1, \ldots, x_{l_i}), \phi_{i,p}(x_1, \ldots, x_{l_i}))$, for $1 \leq p \leq N_i$.

The family of maps $\mathcal{F} := (\psi_{i,k,p})_{i,k,p}$ is a M-resolution, with M depending only on d and $deg(A)$:

- each $\psi_{i,k,p}$ is a Nash triangular map;
- $A = \bigcup_{i,k,p} \psi_{i,k,p}(]0,1[^l)$;
- the cardinal of $\mathcal{F}$ is bounded by $3\sum_{i=1}^m q_i N_i$;
- each $deg(\psi_{i,k,p})$ is bounded by a function of $deg(A)$ and d, according to Corollary 3. ∎

A limit of semi-algebraic maps of bounded degree is again a semi-algebraic map.

Corollary 7: *Let $(f_n :]0,1[^d \to]0,1[^k)_{n\in\mathbb{N}}$ be a sequence of continuous semi-algebraic maps of degree $\leq \delta$, such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f :]0,1[^d \to [0,1]^k$. Then f is a continuous semi-algebraic map of degree bounded by a function of d, k and δ.*

Proof. It is enough to prove the corollary for $k = 1$.

Let $(f_n :]0,1[^d \to]0,1[)_{n\in\mathbb{N}}$ be a sequence of semi-algebraic maps of degree $\leq \delta$. For all $n \in \mathbb{N}$, there exists $P_n \in \mathbb{R}[X_1, \ldots, X_{d+1}] - \{0\}$ of degree $\leq \delta$, such that $P_n(x_1, \ldots, x_d, 1/4 + f_n(x_1, \ldots, x_d)/2) = 0$, $\forall x := (x_1, \ldots, x_d) \in]0,1[^d$. The set $\mathbb{R}[X_1, \ldots, X_{d+1}]$ of polynomials in $d+1$ variables is endowed with the norm: $\|P\| := \sup_{\alpha\in\mathbb{N}^{d+1}} |a_\alpha|$, for $P := \sum_{\alpha\in\mathbb{N}^{d+1}} a_\alpha X^\alpha$. By dividing P_n by $\|P_n\|$, we can choose $\|P_n\| = 1$. Then, by extracting a subsequence, we can assume that $P_n \to P \neq 0$, with $deg(P) \leq \delta$. It is easy to check that $P(x, 1/4 + f(x)/2) = 0$. By applying Proposition 1 to $\{P = 0\} \bigcap]0,1[^{d+1}$ (we consider $1/4 + f_n/2$ instead of f_n, because $f(x)$ might be on the boundary of $[0,1]^d$) and by continuity of f, we conclude there exists a partition of $]0,1[^d$ into Nash manifolds $(A_i)_{i=1,\ldots,N}$ and Nash maps $\zeta_i : A_i \to]0,1[$ with N and $deg(\zeta_i)$ bounded by a function of δ

and d, such that $\Gamma_{1/4+f/2} = \bigcup_{i=1,\dots,N} \Gamma_{\zeta_i}$. In particular, f is a semi-algebraic map o f degree bounded by a function of δ and d. ∎

3.4. $\mathcal{C}^\alpha$-resolution of semi-algebraic sets and Nash maps. In this section, we define notions to estimate the differentiable size of semi-algebraic sets and maps.

Definition 9: A Nash map $f : A \subset \mathbb{R}^d \to \mathbb{R}^n$ is extendable if f extends continuously on adh(A).

Notation 1: Let $f : A \subset \mathbb{R}^d \to \mathbb{R}^n$ be a extendable Nash map. We denote by $\widetilde{f}$ the unique continuous extension of f.

Remark 1: This extension is unique by continuity of f. By using Corollary 1, observe that $\widetilde{f}$ is a semi-algebraic map and that $deg(\widetilde{f})$ is bounded by a function of $deg(f)$.

Definition 10: Let $K \in \mathbb{R}^+$. Let $A \subset]0,1[^d$ be a semi-algebraic set of dimension l. Let $\alpha \in \mathbb{N}^l - \{0\}$. The family of maps $(\phi_i :]0,1[^l \to A)_{i=1,\dots,N}$ is a $(\mathcal{C}^\alpha, K)$-**resolution** of A if:

- each ϕ_i is triangular;
- each ϕ_i is a $(\mathcal{C}^\alpha, K)$ extendable Nash map;
- adh$(A) = \bigcup_{i=1}^N \widetilde{\phi}_i([0,1]^l)$.

Let $M \in \mathbb{N}$. A $(\mathcal{C}^\alpha, K, M)$-**resolution** of A is a $(\mathcal{C}^\alpha, K)$-resolution of A, $(\phi_i)_{i=1,\dots,N}$, such that:

- $N \le M$;
- $deg(\phi_i) \le M$.

Definition 11: Let $K \in \mathbb{R}^+$. Let $f_1, \dots, f_k : A \to]0,1[$ be semi-algebraic maps, where $A \subset]0,1[^d$ is a semi-algebraic set of dimension l. Let $\alpha \in \mathbb{N}^l - \{0\}$. The family of maps $(\phi_i :]0,1[^l \to A)_{i=1,\dots,N}$ is a $(\mathcal{C}^\alpha, K)$-**resolution** of $(f_j)_{j=1,\dots,k}$ if:

- each ϕ_i is triangular;
- each ϕ_i and each $f_j \circ \phi_i$ is a $(\mathcal{C}^\alpha, K)$ extendable Nash map;
- adh$(A) = \bigcup_{i=1}^N \widetilde{\phi}_i([0,1^l])$.

Let $M \in \mathbb{N}$. A $(\mathcal{C}^\alpha, K, M)$-**resolution** of $(f_j)_{j=1,\dots,k}$, $(\phi_i)_{i=1,\dots,N}$, is a $(\mathcal{C}^\alpha, K)$-resolution of $(f_j)_{j=1,\dots,k}$ such that:

- $N \le M$;

- $deg(\phi_i) \leq M$ and $deg(f_j \circ \phi_i) \leq M$.

If $\alpha = (0, 0, \ldots, 0, r)$, we write $(\mathcal{C}^r, K)$, $(\mathcal{C}^r, K, M)$ instead of $(\mathcal{C}^{(0,\ldots,0,r)}, K)$, $(\mathcal{C}^{(0,\ldots,0,r)}, K, M)$.

Remark 2: A $\mathcal{C}^\alpha$-resolution of a semi-algebraic set A is in a obvious way a $\mathcal{C}^\alpha$-resolution of the characteristic function of A.

To prove Yomdin-Gromov's algebraic lemma, we take limits of parametrizations of a semi-algebraic set close to A, so that these limits reparametrize $\mathrm{adh}(A)$. That is why in the definition of a $\mathcal{C}^\alpha$-resolution above we reparametrize $\mathrm{adh}(A)$, contrary to Definition 8 of a resolution.

The following remark is very useful later on:

LEMMA 3: *Given an integer d and a real number N, there is an integer $M = M(N, d)$, such that for any $\alpha \in \mathbb{N}^d - \{0\}$ and for any $(\mathcal{C}^\alpha, N)$ Nash map $f :]0,1[^d \rightarrow]0,1[$, there exists a $(\mathcal{C}^\alpha, 1, M)$-resolution of f.*

Proof. We use homothetic reparametrizations of $]0,1[^d$. The details are left to the reader. ∎

3.5. (α, M)-ADAPTED SEQUENCE. We will use the following notion to prove Yomdin-Gromov's algebraic lemma:

Definition 12: Let $\alpha \in \mathbb{N}^d - \{0\}$ and $M \in \mathbb{N}$. Let $(f_i : A \rightarrow]0,1[)_{i=1,\ldots,k}$ be a family of Nash maps defined on a semi-algebraic open set $A \subset]0,1[^d$. A sequence (α, M)-adapted to $(f_i)_{i=1,\ldots,k}$ is a sequence $(A_n)_{n\in\mathbb{N}}$ of semi-algebraic sets, such that:

- $A_n \subset A$ for each $n \in \mathbb{N}$;
- $a_n := \sup_{x\in A} d(x, A_n) \xrightarrow[n\to+\infty]{} 0$, where $d(x, A_n)$ is the distance between x and A_n;
- $deg(A_n) \leq M$;
- $(f_{i/A_n})_{i=1,\ldots,k}$ admits a $(\mathcal{C}^\alpha, 1, M)$-resolution.

If $\alpha = (0, 0 \ldots, r)$, we write (r, M) instead of $((0, \ldots, 0, r), M)$.

4. Statements

Given a family of semi-algebraic functions, we shall first reparametrize them away from their singularities. Then we prove the main theorem.

Proposition 3: *For any family $(f_i : A \to]0,1[)_{i=1,\dots,k}$ of Nash maps defined on a semi-algebraic open set $A \subset]0,1[^d$, there exists a (r, M)-adapted to $(f_i)_{i=1,\dots,k}$, with M depending only on d, r and $\max_i(deg(f_i))$.*

The next proposition follows from the above:

Proposition 4: *For any family $(f_i : A \to]0,1[)_{i=1,\dots,k}$ of Nash maps defined on a semi-algebraic open set $A \subset]0,1[^d$, there is a $(\mathcal{C}^r, 1, M)$-resolution of $(f_i)_{i=1,\dots,k}$, with M depending only on d, r and $\max_i(deg(f_i))$.*

We deduce the following proposition from Propositions 1 and 4:

Proposition 5: *For any semi-algebraic set $A \subset]0,1[^d$, there exists a $(\mathcal{C}^r, 1, M)$ resolution of A, with M depending only on d, r and $deg(A)$.*

Now we show how Propositions 3, 4 and 5 and Yomdin-Gromov's algebraic lemma follow from the case $k = 1$ of Proposition 3. In fact, we show stronger results, which are used in the induction in the last section.

Notation 2: Let $E = \bigcup_{d\geq 1}(\mathbb{N}^d - \{0\}) \times \{d\}$ together with the order: $(\beta, e) \ll (\alpha, d)$ if $(e < d)$ or $(e = d$ and $\beta \preceq \alpha)$

We write (r, d) instead of $((0, \dots, 0, r), d) \in E$.

The order $\ll$ coincides with the lexicographic order of $(d, |\alpha|, \alpha_d, \dots, \alpha_1)$.

Notation 3: Fix $(\alpha, d) \in E$ and $k \in \mathbb{N}$. We will write $Q3(\alpha, d, k)$, $Q4(\alpha, d, k)$, $Q5(\alpha, d, k)$ for the following claims:

$Q3(\alpha, d, k)$: for any family $(f_i : A \to]0,1[)_{i=1,\dots,k}$ of Nash maps defined on a semi-algebraic open set $A \subset]0,1[^d$, there exists a sequence (α, M)-adapted to $(f_i)_{i=1,\dots,k}$, with $M \in \mathbb{N}$ depending only on $\max_i(deg(f_i))$.

$Q4(\alpha, d, k)$: for any family $(f_i : A \to]0,1[)_{i=1,\dots,k}$ of Nash maps defined on a semi-algebraic open set $A \subset]0,1[^d$, there exists a $(\mathcal{C}^r, 1, M)$-resolution of $(f_i)_{i=1,\dots,k}$, with $M \in \mathbb{N}$ depending only on $\max_i(deg(f_i))$.

$Q5(\alpha, d)$: for any semi-algebraic set $A \subset]0,1[^d$, there exists a $(\mathcal{C}^r, 1, M)$ resolution of A, with $M \in \mathbb{N}$ depending only on $deg(A)$.

In the statements of Propositions 3 and 4, we only need to reparametrize a single Nash map:

Lemma 4: *The claim $Q4(\alpha, d, 1)$ implies the claim $Q4(\alpha, d, k)$ for all $k \in \mathbb{N}^*$.*

Proof. We argue by induction on k. Assume $Q4(\alpha, d, l)$, for $l \leq k$: for any l-families $g_1, \ldots, g_l : B \rightarrow]0,1[$ of Nash maps of degree $\leq \delta$, with $B \subset]0,1[^d$ a semi-algebraic open set, there is a $(\mathcal{C}^\alpha, 1, M)$-resolution of $g_1, \ldots, g_l$, with $M = M(l, \delta)$.

Let $f_1, \ldots, f_{k+1} : A \rightarrow]0,1[$ be Nash maps of degree $\leq \delta$, with $A \subset]0,1[^d$ a semi-algebraic open set. In the following, for each $l \leq k$, we denote $M_l = M(l, \delta)$. According to the induction hypothesis, there exists $(\phi_i)_{i=1,\ldots,N}$ a $(\mathcal{C}^\alpha, 1, M_k)$-resolution of $(f_1, \ldots, f_k)$. By $Q4(\alpha, d)$ for $k = 1$, for each i, we can find $(\psi_{i,j})_{j=1,\ldots,N_i}$ a $(\mathcal{C}^\alpha, 1, M_1)$-resolution of $f_{k+1} \circ \phi_i$.

According to Lemma 2, the maps $\phi_i \circ \psi_{i,j}$, of which the number is $\sum_{i=1}^N N_i \leq M_1 M_k$, are $(\mathcal{C}^\alpha, K)$ extendable Nash maps, with some $K = K(|\alpha|, d)$. The same holds for the maps $f_p \circ \phi_i \circ \psi_{i,j}$ for all $1 \leq p \leq k$. We control the degree of these Nash maps by applying Corollary 3. For each i, $(\psi_{i,j})_{j=1,\ldots,N_i}$ being a $(\mathcal{C}^\alpha, 1)$-resolution of $f_{k+1} \circ \phi_i$, the maps $f_{k+1} \circ \phi_i \circ \psi_{i,j}$ are $(\mathcal{C}^\alpha, 1)$ extendable Nash maps. Moreover, we have in a trivial way: $\mathrm{adh}(A) = \bigcup_{i,j} \widetilde{\phi_i} \circ \widetilde{\psi_{i,j}}([0,1]^d)$. We conclude the proof of $Q4(\alpha, d, k+1)$ by applying Lemma 3. ∎

LEMMA 5: *The claim $Q3(\alpha, d, 1)$ implies the claim $Q3(\alpha, d, k)$ for all $k \in \mathbb{N}^*$.*

Proof. We adapt the above proof for $Q3(\alpha, d)$ as follows (we use the same notation). Let $(A_n)_{n\in\mathbb{N}}$ be a sequence α-adapted to $(f_i)_{i=1,\ldots,k}$. Hence, for all $n \in \mathbb{N}$, there exists $(\phi_j^n)_{j=1,\ldots,N_n}$ a $(\mathcal{C}^\alpha, 1)$ resolution of $(f_{i/A_n})_{i=1,\ldots,k}$. For n, j, let $(A_p^{n,j})_{p\in\mathbb{N}}$ be a sequence α-adapted to $f_{k+1} \circ \phi_j^n$.

We use the following remark, which is an easy consequence of the uniform continuity:

Remark 3: If $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets of $[0,1]^l$ satisfying

$$\sup_{x\in[0,1]^l} d(x, A_n) \xrightarrow[n\to+\infty]{} 0$$

and $\phi : [0,1]^l \to [0,1]^d$ is a continuous map, then

$$\sup_{x\in\phi([0,1]^l)} d(x, \phi(A_n)) \xrightarrow[n\to+\infty]{} 0.$$

According to the above remark for ϕ_j^n, we can choose an integer $p_{j,n}$ for each $n \in \mathbb{N}$ and each $1 \leq j \leq N_n$, such that $\sup_{x\in\phi_j^n([0,1]^d)} d(x, \phi_j^n(A_{p_{j,n}}^{n,j})) < 1/n$. Now, let us show that $B_n := \bigcup_{j=1}^{N_n} \phi_j^n(A_{p_{j,n}}^{n,j})$ defines a sequence α-adapted to $(f_i)_{i=1,\ldots,k+1}$.

Observe that B_n is a semi-algebraic set because each ϕ_j^n is a semi-algebraic map and each $A_p^{n,j}$ is a semi-algebraic set. Moreover, N_n, $deg(\phi_j^n)$ and $deg(A_{p_{j,n}}^{n,j})$ and therefore $deg(B_n)$ are bounded by a function of $\max_i(deg(f_i))$, $|\alpha|$ and d. Finally, we have:

$$\begin{aligned}\sup_{x\in A} d(x,B_n) &\leq \sup_{x\in A} d(x,A_n) + \max_{j=1,\ldots,N_n}\Big(\sup_{x\in\phi_j^n([0,1]^d)} d(x,\phi_j^n(A_{p_{j,n}}^{n,j}))\Big)\\ &\leq a_n + 1/n \xrightarrow[n\to+\infty]{} 0. \qquad \blacksquare\end{aligned}$$

Notation 4: In the following, we note:

$$Qi(\alpha,d) := Qi(\alpha,d,1) = [\forall k\in\mathbb{N}^*,\ Qi(\alpha,d,k)] \quad \text{for } i=3,4 \quad \text{and}$$
$$Pi(\alpha,d) := [\forall(\beta,e)\in E \text{ with } (\beta,e)\ll(\alpha,d),\ Qi(\alpha,d)] \quad \text{for } i=3,4,5.$$

Observe that for $i=3,4,5$, $Pi(\alpha,d)$ is the claim of Proposition i for all pairs $(\beta,e)\in E$ with $(\beta,e)\ll(\alpha,d)$.

Now we show that Proposition 5 follows from Proposition 4:

Proof of Proposition 5 ($P4(r,d)\Rightarrow P5(r,d+1)$). We only need to prove

$$P4(r,d)\Rightarrow Q5(r,d+1).$$

Let $A\subset]0,1[^{d+1}$ be a semi-algebraic set of dimension $l\geq 1$.[3] Under Proposition 1, it is enough to consider the two following special cases:

- $A\subset]0,1[^{d+1}$ is a semi-algebraic set of the form:

  $$\{(x_1,y)\in]0,1[\times A' : \eta(y) < x_1 < \zeta(y)\},$$

  where $A'\subset]0,1[^d$ is a Nash manifold of dimension $l-1$ and $\eta,\zeta : A'\to]0,1[$ are Nash maps, such that $deg(\eta)$, $deg(\zeta)$, $deg(A')$ depend only on $deg(A)$ and d. By using a M-resolution of A', $(\phi_i :]0,1[^{l-1}\to]0,1[^d)_{i=1,\ldots,N}$, with $M=M(d,deg(A))$ and by considering $\eta\circ\phi_i$ and $\zeta\circ\phi_i$, we can assume that $A'=]0,1[^{l-1}$, with $l\leq d+1$. So we can apply $Q4(r,l-1)$ to (ζ,η); there exists $(\phi_i)_{i=1,\ldots,N}$ a $(\mathcal{C}^r,1,M')$-resolution of (ζ,η) with $M'=M'(r,d,deg(A))$. For each i, we define $\psi_i :]0,1[\times]0,1[^{l-1}\to A$ by

  $$\psi_i(x,y) = (x(\zeta\circ\phi_i - \eta\circ\phi_i)(y) + \eta\circ\phi_i(y), \phi_i(y)).$$

[3] The case of dimension 0 is trivial.

We control the degree of ψ_i by applying Corollary 3. Then $(\psi_i)_{i=1,\dots,N}$ is a $(\mathcal{C}^r,2)$-resolution of A. We conclude the proof using Lemma 3.

- A is a semi-algebraic set of the form $\{(\zeta_{i,k}(y),y) : y \in A'\}$. The dimension l of A is strictly less than $d+1$. The decomposition into cells gives us an M-resolution of A, $(\phi_i :]0,1[^l \to A)_{i=1,\dots,N}$, with $M = M(d, deg(A))$. We conclude the proof, by applying for each i, $Q4(r,l)$ to the coordinates of ϕ_i. ∎

Finally, we deduce Proposition 4 from Proposition 3. In fact, we prove: $P3(r+1,d) \Rightarrow P4(r,d)$.

Proof of Proposition 4 ($P3(r+1,d) \Rightarrow P4(r,d)$). We argue by induction on d. Assume that for $e < d$, we have $P3(s+1,e) \Rightarrow P4(s,e)$ for all $s \in \mathbb{N}$. Let $r \in \mathbb{N}$. Let us show $P3(r+1,d) \Rightarrow Q4(r,d)$.

Let $f : A \to]0,1[$ be Nash map of degree $\leq \delta$, where $A \subset]0,1[^d$ is a semi-algebraic open set. According to $Q3(r+1,d)$, there exists a $(r+1,M)$-adapted sequence $(A_n)_{n\in\mathbb{N}}$ to f with $M = M(r,d,\delta)$. Let $(\phi_i^k)_{i\leq N_k}$ be a $(\mathcal{C}^{r+1},1,M)$-resolution of $f_{/A_k}$. For all $k \in \mathbb{N}$, $N_k \leq M$. By extracting a subsequence, we can assume $N_k = N$, for all $k \in \mathbb{N}$. According to the Ascoli theorem, $B(r+1)^{(d+1)N}$ is a compact set in $B(r)^{(d+1)N}$, where $B(r)$ is the closed unit ball of the set of $\mathcal{C}^r$ maps on $]0,1[^d$ onto $\mathbb{R}$, endowed with the norm $\|.\|_r$. By extracting a subsequence from the sequence $(\phi_i^n, f\circ\phi_i^n)_{n\in\mathbb{N}}$, we can assume that for each $i = 1,\dots,N$, $(\phi_i^n)_{n\in\mathbb{N}}$ and $(f\circ\phi_i^n)_{n\in\mathbb{N}}$ converge on $\|\ .\|_r$ norm to $(\mathcal{C}^r,1)$ maps. Let ψ_i be the limit of $(\phi_i^n)_{n\in\mathbb{N}}$. Observe that $f\circ\psi_i = \lim_n f\circ\phi_i^n$ is also a $(\mathcal{C}^r,1)$ map.

By Corollary 7, the maps ψ_i and $f\circ\psi_i$ are semi-algebraic maps of degree bounded by a function depending only on r, δ and d. But a priori, these maps are not Nash maps and they are onto $[0,1]^d$. By applying Corollary 1, we note that $X_i =]0,1[^d - \psi_i^{-1}(\partial[0,1]^d)$ is a semi algebraic set of degree bounded only by a function depending only on r, δ and d.

Let us check that $\bigcup_{i=1,\dots,N}\psi_i(\mathrm{adh}(X_i)) = \mathrm{adh}(A)$. It is enough to show that $A \subset \bigcup_{i=1,\dots,N}\psi_i(\mathrm{adh}(X_i))$, because we have $\psi_i(\mathrm{adh}(X_i)) \subset \mathrm{adh}(A)$, for all i, by convergence of ϕ_i^n to ψ_i. Let $x \in A \subset]0,1[^d$, there exists a sequence $x_n \in A_n \subset]0,1[^d$, such that $x_n \to x$. By extracting a subsequence, we can assume that there exist $1 \leq i \leq N$ and a sequence $(y_n \in [0,1]^d)_{n\in\mathbb{N}}$ such that $x_n = \widetilde{\phi_i^n}(y_n)$. By the uniform convergence of ϕ_i^n to ψ_i, we have $\psi_i(y_n) \to x$. We easily conclude that $\bigcup_{i=1,\dots,N}\psi_i([0,1]^d) = \mathrm{adh}(A)$. But

$[0,1]^d - \mathrm{adh}(X_i) \subset \psi_i^{-1}(\partial[0,1]^d)$; therefore $A \subset \bigcup_{i=1,\dots,N} \psi_i(\mathrm{adh}(X_i))$, because $A \subset]0,1[^d$. Finally, $\mathrm{adh}(A) = \bigcup_{i=1,\dots,N} \psi_i(\mathrm{adh}(X_i))$.

Apply Proposition 1 to the graph $\Gamma_{\psi_{i/X_i}}$ of ψ_{i/X_i}. There exists a partition of X_i into Nash manifold $(X_i^j)_{j=1,\dots,P_i}$, such that ψ_{i/X_i^j} is a Nash map onto $]0,1[^d$. Moreover $P3(r+1,d) \Rightarrow P3(r+1,d-1) \Rightarrow P4(r,d-1) \Rightarrow P5(r,d)$. By applying $P5(r,d)$ to each X_i^j, and by composing the maps ψ_i with the $(\mathcal{C}^r,1)$ Nash map obtained from the $(\mathcal{C}^r,1)$ resolution of X_i^j, we get a $(\mathcal{C}^r,K,M)$-resolution of f, with $K=K(r,d)$ and $M=M(r,deg(f),d)$. We conclude the proof by applying Lemma 3. ∎

Finally, Yomdin-Gromov's algebraic lemma follows from Proposition 5.

Proof of Yomdin-Gromov's algebraic Lemma. Let A be a semi-algebraic compact subset of $[0,1]^d$. We apply $P5(\alpha,d)$ to $A \bigcap F$ for each open hypercube F, which takes part in the skeleton of $[0,1]^d$. ∎

Now we only have to prove Proposition 3 for a single Nash map.

5. Case of dimension 1

First we study the case of dimension 1, where we can prove Proposition 4 right away. The case of dimension 1 allows us to introduce simple ideas of parametrizations, which will be adapted in higher dimensions.

The semi-algebraic sets of $]0,1[$ are the finite unions of open intervals and points. So it is enough to prove Proposition 4 for A of the form $]a,b[\subset]0,1[$. We recall that a bounded Nash map defined on a open bounded interval I extends continuously on $\mathrm{adh}(I)$ (See [7, Proposition 2.3.5]).

Proof of $P4(1,1)$ (Case of the first derivative). Let $f :]a,b[\to]0,1[$ be a Nash map. We cut the interval $]a,b[$ into a minimal number N of subintervals $(J_k)_{k=1,\dots,N}$, such that for each k, $\forall x \in J_k$, $|f'(x)| \geq 1$ or $\forall x \in J_k$, $|f'(x)| \leq 1$. The integer N is bounded by a function of $deg(f)$: apply Proposition 1 to $\{x \in]0,1[\ |f'(x)| \leq 1\}$ and $\{x \in]0,1[\ |f'(x)| \geq 1\}$ and use Corollary 4.

On each interval J_k, we consider the following parametrization ϕ of $\mathrm{adh}(J_k) = [c,d] \subset [0,1]$:

- $\phi(t) = c + t(d-c)$ if $|f'| \leq 1$, and then we have $deg(\phi) = 1$, $deg(f \circ \phi) = deg(f)$.

- $\phi(t) = f^{-1}_{|[c,d]}(f(c)+t(f(d)-f(c)))$ if $|f'| \geq 1$, and then we have $deg(\phi) = deg(f)$ (indeed $deg(f^{-1}) = deg(f)$) and $deg(f \circ \phi) = 1$. ∎

Proof of $P4(r,1)$ *(Case of higher derivatives).* We argue by induction on r. Assume $P4(r,1)$, with $r \geq 1$ and prove $P4(r+1,1)$.

Let $f :]a,b[\subset]0,1[\to]0,1[$ be a Nash map. By considering for all $i = 1,\ldots,N$ the family $(f \circ \phi_i, \phi_i)$, where $(\phi_i)_{i=1,\ldots N}$ is a $(\mathcal{C}^r,1,M)$ resolution of f (with $M = M(r)$) given by $P4(r,1)$, we can assume that f is a $(C^r,1)$ Nash map.

We divide the interval $]a,b[$ into a minimal number N of subintervals on which $|f^{(r+1)}|$ is either increasing or decreasing, i.e., the sign of $f^{(r+1)}f^{(r+2)}$ is constant. Consider the case where $|f^{(r+1)}|$ is decreasing, the increasing case being similar. We reparametrize these intervals from $[0,1]$ with linear increasing maps Φ_i. We define $f_i = f \circ \Phi_i$. Obviously f_i is a $(C^r,1)$ Nash map and $|f_i^{(r+1)}|$ is decreasing. In the following computations, we note f instead of f_i.

Setting $h(x) = x^2$, we have:

$$(f \circ h)^{(r+1)}(x) = (2x)^{r+1}f^{(r+1)}(x^2) + R(x, f(x), \ldots, f^{(r)}(x))$$

where R is a polynomial depending only on r. Therefore,

$$\forall x \in]0,1[\quad |(f \circ h)^{(r+1)}(x)| \leq |(2x)^{r+1}f^{(r+1)}(x^2)| + C(r), \tag{1}$$

where $C(r)$ is a function of r.

Furthermore, we have

$$x|f^{(r+1)}(x)| = \int_0^x |f^{(r+1)}(x)|dt \leq \left|\int_0^x f^{(r+1)}(t)dt\right| = |f^{(r)}(x) - f^{(r)}(0)| \leq 2. \tag{2}$$

Indeed, either $f^{(r+1)}(x) = 0$ and then the inequality is trivial or $f^{(r+1)}(x) \neq 0$ and therefore the sign of $f^{(r+1)}(t)$ is constant because $|f^{(r+1)}|$ being decreasing, we have for $t \in]0,x]$: $0 < |f^{(r+1)}(x)| \leq |f^{(r+1)}(t)|$. By combining inequalities (1) and (2), we obtain:

$$|(f \circ h)^{(r+1)}(x)| \leq C(r) + 2\frac{(2x)^{r+1}}{x^2} \leq C(r) + 2^{r+2}$$

Finally $deg(\Phi_i \circ h) = 2$ and $deg(f \circ h) = 2deg(f)$. We show now that N is bounded by a function of $deg(f)$ and r like in the first step of the proof: we apply Proposition 1 to the semi-algebraic set $\{x \in]0,1[: f^{(r+1)}(x)f^{(r+2)}(x) \geq 0\}$ and we use Corollary 4.

We conclude the proof of $P4(r+1,1)$ by applying Lemma 3. ∎

6. Proof of Proposition 3

The proof of Proposition 3 is an induction both on the dimension d and on the order of derivation α.

In the first step we increase the dimension d.

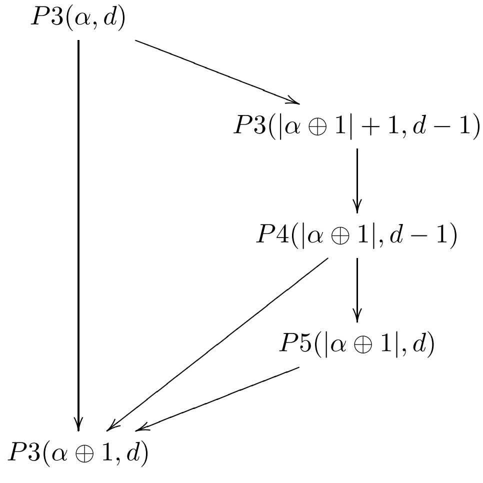
Then, fixing the dimension d, we increase the order of derivation α according to the total order $\preceq$. To be more explicit, let us introduce the following notation:

Notation 5: For $\alpha \in \mathbb{N}^d$, we set:

$$\alpha \oplus 1 := \min\{\beta \in \mathbb{N}^d : \alpha \preceq \beta \ \ \textit{and}\ \alpha \neq \beta\}$$

We prove $P3(\alpha, d) \Rightarrow P3(\alpha \oplus 1, d)$. We will consider two cases: $|\alpha \oplus 1| = |\alpha| + 1$, i.e., $\alpha = (0, \ldots, 0, s)$, for some $s \in \mathbb{N}$ and $|\alpha \oplus 1| = |\alpha|$.

In fact, we prove in this section Yomdin-Gromov's Lemma by induction. We summarize in the following diagram the different dependences involved in the induction:



Increase of the dimension: $[\forall r \in \mathbb{N}\ P3(r, d)] \Rightarrow P3((1, 0, \ldots, 0), d+1)$

Proof. Let $f : A \subset]0,1[^{d+1} \to]0,1[$ a Nash map, defined on a semi-algebraic open set $A \subset \mathbb{R}^{d+1}$. We work on $A_n = \{x \in A : d(x, A_n^c) > 1/n\}$ in order to ensure that f extends continuously on $\mathrm{adh}(A_n)$. The set A_n is a semi-algebraic open set of degree bounded by a function of $deg(A)$ and d (Corollary 1). For simplicity, we note A instead of A_n.

We consider the following semi-algebraic open sets:

$$A_+ = \{x \in A, |\partial_{x_1} f(x)| > 1\} \quad \text{and} \quad A_- = \text{int}(\{x \in A, |\partial_{x_1} f(x)| \leq 1\}).$$

We have $\text{adh}(A) = \text{adh}(A_+) \bigcup \text{adh}(A_-)$. Obviously $\text{adh}(A_+) \bigcup \text{adh}(A_-) \subset \text{adh}(A)$. Let us show $A \subset \text{adh}(A_+) \bigcup \text{adh}(A_-)$. Let $y \in A$, if $y \notin \text{adh}(A_+)$, as A is open, there exists $r > 0$, such that the ball $B(y,r) \subset A \bigcap A_+^c \subset \{x \in A, |\partial_{x_1} f(x)| \leq 1\}$ and thus $y \in A_-$. Remark that $deg(A_+), deg(A_-)$ are bounded by a function of $deg(f)$ according to Corollary 1.

According to $P3(2,d) \Rightarrow P4(1,d) \Rightarrow P5(1,d+1)$, there exist $(\mathcal{C}^1, 1)$ extendable Nash triangular maps $(\phi_j)_{1\leq j\leq N}$, such that $\text{adh}(A_-) = \bigcup_{1\leq j\leq N_-} \widetilde{\phi_j}([0,1]^d)$ and such that N_-, $deg(\phi_j)$ are bounded by a function of $deg(A_-)$, and thus by a function of $deg(f)$. We have $|\partial_{x_1}(f \circ \phi_j)| \leq 1$, so the maps ϕ_i can be used to build a resolution of f.

For A_+, we consider the inverse of f. Observe first, that according to Corollary 5, we can assume that A_+ is a slice of the following form $\{(x_1, y) \in]0,1[\times A'_+ : \zeta(y) < x_1 < \eta(y)\}$, where $A'_+ \subset]0,1[^d$ is a semi-algebraic open set of $\mathbb{R}^d$ and $\zeta, \eta : A'_+ \to]0,1[$ are Nash maps.

Define $D_+ = \{(f(x_1,y), y) : (x_1,y) \in A_+\}$. We define $g : A_+ \to D_+$, $g(x_1,y_1) := (f(x_1,y),y))$. The map g is a local diffeomorphism, by the local inversion theorem. Moreover, g is one to one, because $g(x_1,y) = g(x'_1,y')$ implies $y = y'$, and $f(x_1,y) = f(x'_1,y)$ implies $x_1 = x'_1$, because $|\partial_{x_1} f(x)| \geq 1$ for $x \in A_+$. The map g extends to a homeomorphism $g : \text{adh}(A_+) \to \text{adh}(D_+)$, since f is continuous on $adh(A)$ (recall that we denote $A := A_n$).

Observe that D_+ is a semi-algebraic open set of $\mathbb{R}^{d+1}$. On D_+ we define ϕ: $\phi(t,u) := g^{-1}(t,u) = (f(.,u)^{-1}(t), u)$. The Nash map $\phi : D_+ \to A_+$ is triangular and $deg(\phi) = deg(f)$. Define $\phi(t,u) = (x_1, y)$. We compute:

$$D\phi(t,u) = \begin{pmatrix} \frac{1}{\partial_{x_1} f(x_1,y)} & -\frac{1}{\partial_{x_1} f} \nabla_y f(x_1,y) \\ 0 & Id \end{pmatrix}.$$

As $(x_1,y) \in A_+$, we have $|\partial_{x_1}\phi(t,u)| = |\frac{1}{\partial_{x_1} f(x_1,y)}| \leq 1$. Furthermore, we check

$$f \circ \phi(t,u) = t.$$

Therefore, ϕ and $f \circ \phi$ are $(\mathcal{C}^{(1,0,\ldots,0)}, 1)$ extendable Nash triangular maps. In order to obtain a resolution, we apply again $P5(1,d+1)$ to D_+. That gives $(C^1, 1)$ extendable Nash triangular maps $\psi_j :]0,1[^{d+1} \to D_+$, $j \leq N_+$, such that N_+, $deg(\psi_j)$ are bounded by a function of $deg(D_+)$, thus by a function of

$deg(f)$ and such that $\mathrm{adh}(D_+) = \bigcup_{1\leq i\leq N_+} \widetilde{\psi_i}([0,1]^{d+1})$. Moreover, by applying Fact 4, we get:

$$|\partial_{x_1}(\phi \circ \psi_j)| = |\partial_{x_1}(\phi)|.|\partial_{x_1}(\psi_j^1)| \leq 1$$

because ψ_j is triangular. We also have:

$$|\partial_{x_1}(f \circ \phi \circ \psi_j)| = |\partial_{x_1}\psi_j^1| \leq 1,$$

where ψ_j^1 is the first coordinate of ψ_j. The parametrizations $\phi \circ \psi_j :]0,1[^{d+1} \mapsto]0,1[^{d+1}$ are therefore $(\mathcal{C}^{(1,0,\ldots,0)}, 1)$ extendable Nash triangular maps, such that:

- $\mathrm{adh}(A_+) = \bigcup_{j=1}^{N_+} \widetilde{\phi} \circ \widetilde{\psi_j}([0,1]^{d+1})$;
- each $f \circ \phi \circ \psi_j$ is a $(\mathcal{C}^{(1,0,\ldots,0)}, 1)$ Nash map;
- $deg(\phi \circ \psi_j)$, $deg(f \circ \phi \circ \psi_j)$ are bounded by a function of $|\alpha|, d$, and $deg(f)$ according to Corollary 3.

Finally, $\{\phi_1, \ldots, \phi_{N_-}, \phi \circ \psi_1, \ldots, \phi \circ \psi_{N_+}\}$ is a $(\mathcal{C}^{(1,0,\ldots,0)}, 1)$-resolution of f. ∎

Increase of the derivation order: $P3(s,d) \Rightarrow P3((s+1,0,\ldots,0),d)$.

Proof. Like in the case of dimension $d = 1$, we begin with the following reduction.

Claim 1: *It is enough to show the result for a single $(\mathcal{C}^s, 1)$ extendable Nash map $f : A =]0,1[^d \to]0,1[$.*

Proof of Claim 1. Assume that $P3((s+1,0,\ldots,0),d)$ for a single $(\mathcal{C}^s, 1)$ extendable a Nash map $f : A =]0,1[^d \to]0,1[$. The proof of Lemma 5 implies $P3((s+1,0,\ldots,0),d)$ for any family $(g_i :]0,1[^d \to]0,1[)_{i=1,\ldots,k}$ of $(\mathcal{C}^s, 1)$ extendable Nash maps. Let $f : A \subset]0,1[^d \to]0,1[$ a Nash map, defined on a semi-algebraic open set $A \subset]0,1[^1$. By applying $Q3(s,d)$ to f, we obtain a $(\mathcal{C}^s, 1)$ resolution $(\phi_i^n)_{i=1,\ldots,N_n}$ of $f_{/A_n}$, with A_n an adapted sequence. We apply $Q3((s+1,0,\ldots,0),d)$ to the family $(f \circ \phi_i^n, \phi_i^n)$ of $(\mathcal{C}^s, 1)$ extendable Nash maps defined on $]0,1[^d$. We conclude by constructing a $((s+1,0,\ldots,0),M)$-adapted sequence for f with $M = M(s,d,deg(f))$, like in the proof of Lemma 5. ∎

Let $f :]0,1[^d \to]0,1[$ be a $(\mathcal{C}^s, 1)$ Nash map.

We cut up $]0,1[^d$ according to the sign of $\frac{\partial^{s+1}f}{\partial x_1^{s+1}}\frac{\partial^{s+2}f}{\partial x_1^{s+2}}$ like in the first step of the proof:

$$A_+ = \Big\{x \in]0,1[^d, \frac{\partial^{s+1}f}{\partial x_1^{s+1}}(x)\frac{\partial^{s+2}f}{\partial x_1^{s+2}}(x) > 0\Big\}$$

and

$$A_- = \operatorname{int}\Big(\Big\{x \in]0,1[^d, \frac{\partial^{s+1}f}{\partial x_1^{s+1}}(x)\frac{\partial^{s+2}f}{\partial x_1^{s+2}}(x) \leq 0\Big\}\Big).$$

We have again $\operatorname{adh}(A) = \operatorname{adh}(A_+)\bigcup \operatorname{adh}(A_-)$. In the following, we consider only $A = A_+$, the case of A_- being similar. According to Corollary 5, we can assume that A is a slice of the following form $\{(x_1,y) \in]0,1[\times A'\; \zeta(y)<x_1<\eta(y)\}$, where $A' \subset]0,1[^{d-1}$ is a semi-algebraic open set and $\zeta,\eta : A' \to]0,1[$ are Nash maps.

Applying estimate (2) obtained in Part 5 to the function $x_1 \mapsto \frac{\partial^{s+1}f}{\partial x_1^{s+1}}(x_1,y)$ (we fix y), we get for $(x_1,y)\in A_+$,

$$\Big|\frac{\partial^{s+1}f}{\partial x_1^{s+1}}(x_1,y)\Big| \leq \frac{2}{|x_1-\zeta(y)|}. \tag{3}$$

The induction hypothesis $P3(s,d)$ implies $P3(s+2,d-1)$ and $P3((s+2,d-1)$ implies $P4(s+1,d-1)$. Apply $P4(s+1,d-1)$ to (ζ,η): there exist $(\mathcal{C}^{s+1},d-1)$ extendable Nash triangular maps $h :]0,1[^{d-1}\to]0,1[^{d-1}$, of which the images of the extensions cover $\operatorname{adh}(A')$, such that $\zeta\circ h$ and $\eta\circ h$ are $(\mathcal{C}^{s+1},d-1)$ Nash maps. Define $\psi :]0,1[\times]0,1[^{d-1}\to A$,

$$\psi(v_1,w) = (\zeta\circ h(w).(1-v_1^2)+\eta\circ h(w).v_1^2, h(w)).$$

The maps ψ are triangular, $\|\psi\|_{s+1}\leq 2$ and the images of their continuous extensions cover $\operatorname{adh}(A)$.

In the new coordinates $(v_1,v_2,\dots,v_d) =: (v_1,w)$, we have:

$$x_1-\zeta(y) = \zeta\circ h(w).(1-v_1^2)+\eta\circ h(w).v_1^2-\zeta(h(w)) = v_1^2.(\eta\circ h(w)-\zeta\circ h(w))$$

and therefore the previous estimate (3) becomes:

$$\Big|\frac{\partial^{s+1}f}{\partial x_1^{s+1}}(\psi(v_1,w))\Big| \leq \frac{2}{v_1^2|\eta\circ h(w)-\zeta\circ h(w)|}. \tag{4}$$

Moreover, applying Fact 4, we get:

$$\begin{aligned}\frac{\partial^{s+1}(f\circ\psi)}{\partial v_1^{s+1}}(v_1,w) =&(2v_1)^{s+1}(\eta\circ h(w)-\zeta\circ h(w))\frac{\partial^{s+1}f}{\partial x_1^{s+1}}(\psi(v_1,w))\\ &+R(\eta\circ h(w)-\zeta\circ h(w),v_1,(\frac{\partial^k f}{\partial x_1^k}(\psi(v_1,w)))_{k\le s}),\end{aligned}$$

where R is a polynomial, which depends only on s and d. Using the last inequality (4), the first term is less than 2^{s+2}. Consider the second term. The map f is a $(\mathcal{C}^s,1)$ Nash map, therefore, $|\frac{\partial^k f}{\partial x_1^k}|\le 1$, for $k\le s$. Thus $\left|R(\eta\circ h(w)-\zeta\circ h(w),v_1,\left(\frac{\partial^k f}{\partial x_1^k}(\psi(v_1,w))\right)_{k\le s})\right|$ is bounded by a function of s and d, and hence $\left|\frac{\partial^{s+1}(f\circ\psi)}{\partial v_1^{s+1}}\right|$ also. We apply Lemma 1 to control the derivatives of lower order than s of $f\circ\psi$. Using Lemma 3, we can assume that ψ is a $(\mathcal{C}^{s+1},1)$ Nash map and $f\circ\psi$ is a $(\mathcal{C}^{(s+1,0,\ldots,0)},1)$ Nash map. ∎

We deal now with the last step of the proof.

Control of the following derivative: $P3(\alpha,d)\Rightarrow P3(\alpha\oplus 1,d)$ with $\alpha\neq(0,\ldots,0,s)$

Observe that the condition $\alpha\neq(0,\ldots,0,s)$ implies $|\alpha|=|\alpha\oplus 1|$: the order of the derivation is fixed.

Proof. Like in Claim 1, we can assume that $f:]0,1[^d\to]0,1[$ is a $(\mathcal{C}^\alpha,1)$ Nash map.

Define $A_n=]1/n,1-1/n[^{d-1}$. According to Tarski's principle,

$$B=\left\{(x_1,y)\in \mathrm{adh}(A_n):\ \left|\frac{\partial^{\alpha\oplus 1}f}{\partial x^{\alpha\oplus 1}}(x_1,y)\right|=\sup_{t\in[1/n,1-1/n]}\left(\left|\frac{\partial^{\alpha\oplus 1}f}{\partial x^{\alpha\oplus 1}}(t,y)\right|\right)\right\}$$

is a semi-algebraic set of degree bounded by a function of $deg(f)$ and s. By the definition of an adapted sequence, the sup above is finite (recall that f is not supposed to be analytic in a neighborhood of A, so that we cannot work directly with A). According to Proposition 1, B is covered by sets $(B_i)_{i=1,\ldots,N}$, $B_i=\{(x_1,y)\in]0,1[\times B_i':\gamma_i(y)<x_1<\Delta_i(y)\}$ or $B_i=\{(\sigma_i(y),y)\in B_i'\}$, where $B_i'\subset]1/n,1-1/n[^{d-1}$ are semi-algebraic sets of $\mathbb{R}^{d-1}$, such that $\bigcup_{i=1}^N B_i'=]1/n,1-1/n[^{d-1}$ and where $\sigma_i,\gamma_i,\Delta_i:B_i'\to]0,1[$ are Nash maps. In the first case, we set $\sigma_i:=1/2(\Delta_i+\gamma_i)$. Afterwards, we consider only the sets B_i', which are open sets. Observe that for these sets we have $\bigcup \mathrm{adh}(B_i')=[1/n,1-1/n]^{d-1}$.

By using the Tarski's principle and Proposition 1, we check that N and the degree of σ_i are bounded by a function of $deg(f)$ and $|\alpha|$. Define the Nash map $g_i : B'_i \to]0,1[$, $g_i(y) = \frac{1}{2}\frac{\partial^{(\alpha\oplus 1)_1} f}{\partial x_1^{(\alpha\oplus 1)_1}}(\sigma_i(y), y)$, where $(\alpha\oplus 1)_i$ denotes the i-th coordinate of $\alpha\oplus 1$. The map g_i is onto $]0,1[$, because f is a $(\mathcal{C}^\alpha, 1)$ map and $((\alpha\oplus 1)_1, 0, \dots, 0) \preceq \alpha$. The induction hypothesis $P3(\alpha, d)$ implies $P3(|\alpha|+1, d-1)$ and thus $P4(|\alpha|, d-1)$, which applied to (σ_i, g_i) gives $(\mathcal{C}^{|\alpha|}, 1)$ extendable Nash triangular maps $h_{i,k} :]0,1[^{d-1} \to B'_i$, such that $g_i \circ h_{i,k}$ and $\sigma_i \circ h_{i,k}$ are $(\mathcal{C}^{|\alpha|}, 1)$ Nash and such that $\bigcup_k \widetilde{h_{i,k}}([0,1]^{d-1}) = adh(B'_i)$.

Then, $h_{i,k}$ being triangular, we have according to Fact 4:

$$2\frac{\partial^{((\alpha\oplus 1)_2,\dots,(\alpha\oplus 1)_d)}(g_i\circ h_{i,k})}{\partial x^{((\alpha\oplus 1)_2,\dots,(\alpha\oplus 1)_d)}}(y) = \frac{\partial^{\alpha\oplus 1} f}{\partial x^{\alpha\oplus 1}}(\sigma_i \circ h_{i,k}(y), h_{i,k}(y)) \times \Big(\frac{\partial h_{i,k}}{\partial_{x_2}}\Big)^{(\alpha\oplus 1)_2} \cdots \Big(\frac{\partial h_{i,k}}{\partial x_d}\Big)^{(\alpha\oplus 1)_d} + R$$

where R is a polynomial, depending only on α, in the derivatives of f of order $\preceq \alpha$ and in the derivatives of $h_{i,k}$ and $\sigma_i \circ h_{i,k}$ of order less than $|\alpha|$. The map $h_{i,k}$ is a $(\mathcal{C}^{|\alpha|}, 1)$ Nash map and by hypothesis f is a $(\mathcal{C}^\alpha, 1)$ Nash map, so that we have $|R| < C(|\alpha|, d)$, where C is a function [4] of $|\alpha|$ and d. After all $g_i \circ h_{i,k}$ is a $(\mathcal{C}^{|\alpha|}, 1)$ Nash map. Hence,

$$\Big|\frac{\partial^{\alpha\oplus 1} f}{\partial x^{\alpha\oplus 1}}(\sigma_i \circ h_{i,k}(y), h_{i,k}(y))\Big(\frac{\partial h_{i,k}}{\partial_{x_2}}\Big)^{(\alpha\oplus 1)_2} \cdots \Big(\frac{\partial h_{i,k}}{\partial x_d}\Big)^{(\alpha\oplus 1)_d}\Big| \leq \Big|2\frac{\partial^{((\alpha\oplus 1)_2,\dots,(\alpha\oplus 1)_d)}(g_i\circ h_{i,k})}{\partial x^{((\alpha\oplus 1)_2,\dots,(\alpha\oplus 1)_d}}\Big| + |R| < C(|\alpha|, d)$$

Define $\phi_{i,k} :]0,1[^d \to]0,1[^d$ by:

$$\phi_{i,k}(x_1, y) = (1/n + b_n x_1, h_{i,k}(y)), \quad \text{with} \quad b_n := 1 - 2/n.$$

The parametrization $\phi_{i,k}$ is a $(\mathcal{C}^{\alpha\oplus 1}, 1)$ Nash triangular map:

- Using again the triangularity of $h_{i,k}$ and Fact 4, we get: $\frac{\partial^{\alpha\oplus 1}(f\circ\phi_{i,k})}{\partial x^{\alpha\oplus 1}} = \frac{\partial^{\alpha\oplus 1} f}{\partial x^{\alpha\oplus 1}}(1/n + b_n x_1, h_{i,k}(y)) \times (b_n)^{(\alpha\oplus 1)_1}(\frac{\partial h_{i,k}}{\partial_{x_2}})^{(\alpha\oplus 1)_2} \dots (\frac{\partial h_{i,k}}{\partial x_d})^{(\alpha\oplus 1)_d} + S$, where S is a polynomial in $\frac{\partial^\beta f}{\partial x^\beta}$ with $\beta \preceq \alpha$ and in the derivatives of $h_{i,k}$ of order less than $|\alpha|$, S depending only on α. Threfore $|S| < C(|\alpha|, d)$.

[4] In order to simplify the notations, we will denote by C any function of $|\alpha|$ and d

Moreover, by the definition of σ_i,

$$\begin{aligned}&\Big|\frac{\partial^{\alpha\oplus 1} f}{\partial x^{\alpha\oplus 1}}(1/n+b_n x_1, h_{i,k}(y)) \times \Big(\frac{\partial h_{i,k}}{\partial_{x_2}}\Big)^{(\alpha\oplus 1)_2} \dots \Big(\frac{\partial h_{i,k}}{\partial x_d}\Big)^{(\alpha\oplus 1)_d}\Big| \\ &\leq \Big|\frac{\partial^{\alpha\oplus 1} f}{\partial x^{\alpha\oplus 1}}(\sigma_i \circ h_{i,k}(y), h_{i,k}(y)) \times \Big(\frac{\partial h_{i,k}}{\partial_{x_2}}\Big)^{(\alpha\oplus 1)_2} \dots \Big(\frac{\partial h_{i,k}}{\partial x_d}\Big)^{(\alpha\oplus 1)_d}\Big| \\ &< C(|\alpha|, d),\end{aligned}$$

thus

$$\begin{aligned}&\Big|\frac{\partial^{\alpha\oplus 1}(f\circ \phi_{i,k})}{\partial x^{\alpha\oplus 1}}\Big| \\ &\leq \Big|\frac{\partial^{\alpha\oplus 1} f}{\partial x^{\alpha\oplus 1}}(1/n+b_n x_1, h_{i,k}(y)) \times \Big(\frac{\partial h_{i,k}}{\partial_{x_2}}\Big)^{(\alpha\oplus 1)_2} \dots \Big(\frac{\partial h_{i,k}}{\partial x_d}\Big)^{(\alpha\oplus 1)_d}\Big| + |S| \\ &< C(|\alpha|, d)\end{aligned}$$

- Finally for $\beta \preceq \alpha$, only the derivatives of f of order $\preceq \alpha$ take part in the expression $\frac{\partial^\beta (f\circ\phi_{i,k})}{\partial x^\beta}$, again because of the triangularity of $h_{i,k}$ and Fact 4. Hence $\big|\frac{\partial^\beta (f\circ\phi_{i,k})}{\partial x^\beta}\big| < C(|\alpha|, d)$.

Lemma 3 gives us a $(\mathcal{C}^\alpha, 1, M)$-resolution of $f_{/A_n}$, with

$$M = M(|\alpha|, d, deg(f)). \qquad \blacksquare$$

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